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Non-classical symmetry reduction: example of the Boussinesq equation

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Abstract. A symmetry of an equation will leave the set of all solutions invariant. A 'conditional symmetry' will leave only a subset of solutions, defined by some differential condition, invariant. We show how a specific class of conditional symmetries can be used to reduce a partial differential equation to an ordinary one. In particular, for the Boussinesq equation, these conditional symmetries, together with the ordinary ones, provide all possible reductions to ordinary differential equations. A group theoretical explanation of the recently obtained new reductions is provided.

1. Introduction

A standard way of obtaining particular solutions of non-linear partial differential equations (PDE) is the method of symmetry reduction. The essence of the method is that one looks for solutions of the equation that are invariant under some chosen subgroup G_0 of the symmetry group G of the equation. This invariance makes it possible to rewrite the equation in terms of group invariants and thus to reduce the number of independent variables involved. If certain technical conditions on the structure of the orbits of G_0 (acting on the space of independent and dependent variables) are satisfied, then the equation can be reduced to an ordinary differential equation [1-3] (ODE). The method is entirely algorithmic and can, to a large degree, be performed on a computer, using various symbol manipulating packages [4, 5].

Recently Clarkson and Kruskal [6] (hereafter referred to as CK) have studied the Boussinesq equation

$$u_{tt} + uu_{xx} + (u_x)^2 + u_{xxxx} = 0 \tag{1.1}$$

and found all of its similarity solutions. By this they mean all solutions of the form

$$u(x, t) = U(x, t, w(z)) \qquad z = z(x, t) \tag{1.2}$$

where U and z are functions of the indicated variables and $w(z)$ satisfies an ordinary differential equation (ODE), obtained by substituting (1.2) into (1.1).

Their method was entirely straightforward, made no use of group theory and led to some complicated equations for the variable $z(x, t)$ and the function U of (1.2). In a *tour de force* the authors solved the equations and obtained numerous new reductions of the Boussinesq equation (1.1) (in addition to the known ones, due to dilational and

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translational invariance). They expressed the 'hope that a group theoretic explanation of the method will be possible in due course'.

The purpose of this paper is to provide such an explanation and, by the same token, to advocate an algorithmic procedure for reducing partial differential equations to ordinary ones, going beyond classical symmetry reduction. The framework is one that is already available, namely the 'non-classical method' introduced by Bluman and Cole [7].

In § 2 we reformulate and reinterpret this non-classical method in a manner that is well adapted to existing computer programs and elucidate some of its features. In particular, we stress that 'non-classical symmetries' are not symmetries of the equation itself, but of the equation, together with a specific auxiliary condition. This first-order PDE plays the role of the 'side condition' introduced by Olver and Rosenau [8, 9]. In the same section we apply this non-classical method and the corresponding 'conditional symmetries' to the Boussinesq equation. In a quite simple manner we retrieve the general structure of the CK results. The individual CK reductions are all obtained in § 3, together with a group theoretical explanation. Section 4 is devoted to conclusions and open problems.

2. General form of the conditional symmetries and non-classical reductions

2.1. General comments

Let us first reformulate the Bluman and Cole 'non-classical method' [7] in terms of vector fields and their prolongations. This could be done for an arbitrary system of differential equations, but for the purposes of this paper, we restrict ourselves to one nth-order PDE for a function $u(x, t)$ of two variables. We write this equation formally as

$$\Delta^n(x, t, u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0 \tag{2.1}$$

(where the subscripts on u denote partial derivatives).

The Lie algebra of the Lie group of local point transformations leaving (2.1) invariant consists of vector fields of the form

$$V = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \phi(x, t, u)\partial_u \tag{2.2}$$

where ξ, τ and ϕ are functions of x, t and u that are determined from the invariance requirement

$$\mathrm{pr}^{(n)} V \cdot \Delta^{(n)}|_{\Delta^{(n)}=0} = 0. \tag{2.3}$$

The nth prolongation $\mathrm{pr}^{(n)}V$ of V is given explicitly in terms of ξ, τ and ϕ of (2.2), e.g. in [3], and we do not reproduce it here.

Instead of looking for this 'symmetry algebra' in the classical sense, let us leave V undetermined and let us add an auxiliary first-order equation to (2.1), namely

$$\Delta^{(1)}(x, t, u, u_t, u_x) = \xi(x, t, u)u_x + \tau(x, t, u)u_t - \phi(x, t, u) = 0. \tag{2.4}$$

Equation (2.4) is as yet unspecified and it will be determined together with the vector field V, involving the same functions ξ, τ and ϕ.

We now look for the simultaneous symmetry group of equations (2.1) and (2.4), using the classical method. In other words we require that the appropriate prolongation of the vector field should annihilate both equations on the solution surface of both equations:

$$\mathrm{pr}^{(n)}V\cdot \Delta^{(n)}|_{\Delta^{(n)}=0,\,\Delta^{(1)}=0}=0 \tag{2.5}$$

$$\mathrm{pr}^{(1)}V\cdot \Delta^{(1)}|_{\Delta^{(n)}=0,\,\Delta^{(1)}=0}=0. \tag{2.6}$$

Notice that condition (2.6) is satisfied trivially and is hence no restriction on V. Indeed (2.4) was chosen precisely because $\Delta^{(1)}$ is an invariant of V, i.e. condition (2.6) is an identity, satisfied for all functions ξ, τ and ϕ. Indeed, we have

$$\mathrm{pr}^{(1)}V\cdot \Delta^{(1)}=-(\xi_u u_x+\tau_u u_t-\phi_u)\Delta^{(1)} \tag{2.7}$$

which vanishes for $\Delta^{(1)}=0$, without imposing any conditions on the functions ξ, τ and ϕ.

We can assume $\tau(x,t,u)\neq 0$ in (2.4) and use (2.4), together with its differential consequences, to eliminate u_t and all higher derivatives involving time (u_{tt}, u_{tx}, u_{txx}, ...) from equation (2.5). The highest derivative involving only the x variable is eliminated from (2.5) using (2.1). The coefficients of the linearly independent expressions in the remaining derivatives must then be set equal to zero. This provides a system of determining equations for the functions $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. In general, these equations will be non-linear.

If we can solve the determining equations we obtain the explicit form of the vector field V, which we shall call a 'conditional symmetry' operator. If we integrate such a vector field we obtain a one-parameter group of local point transformations:

$$x'=X_\lambda(x,t,u) \qquad t'=T_\lambda(x,t,u) \qquad u'=U_\lambda(x,t,u). \tag{2.8}$$

These transformations will, however, in general not leave (2.1) invariant, i.e. they will not transform solutions amongst each other. What they will do, is take solutions of (2.1), that also satisfy the condition (2.4), into solutions that also satisfy both equations (hence 'conditional symmetries').

Thus, conditional symmetries are of no great use for obtaining new solutions from old ones. On the other hand, they turn out to be extremely useful for performing symmetry reduction. Indeed, we can look for solutions of the considered equation (2.1) that are invariant under the transformations (2.8). This boils down to finding the two invariants of (2.8) (directly from the conditional symmetry operator V), say

$$I_1(x,t)\equiv\xi \qquad I_2(x,t,u)=F. \tag{2.9}$$

Just as in the classical method we can then express $u(x,t)$ in terms of F (if $\partial I_2/\partial_u\neq 0$), consider F as a function of ξ, substitute back into (2.1) and obtain an ODE for $F(\xi)$. We are actually performing a classical reduction for the studied equation and the supplementary condition (2.4), but that auxiliary equation is already solved trivially, and hence imposes no further restrictions.

Several comments on conditional symmetries are in order here. First of all, these symmetries do not form a vector space, still less a Lie algebra. Indeed, each conditional symmetry operator is adapted to its own auxiliary equation (2.4). Moreover, the auxiliary equations are in general not invariant under the symmetry group. Hence a linear combination of an ordinary (classical) symmetry and a conditional symmetry is, in general, not a symmetry at all (and will not provide a reduction of the PDE).

On the other hand, the symmetry group G of a PDE can be put to good use in connection with the conditional symmetries. Indeed, once conditional symmetry operators are found, they can be simplified and classified into conjugacy classes under the action of the group G. A representative of each conjugacy class will give a specific reduction. The other reductions belonging to the same class will then be re-obtained by group transformations. Once a solution $u(x, t)$ is found, the entire group G can be applied to it, since we no longer need to impose the supplementary condition (2.4).

2.2. General procedure for the Boussinesq equation

We shall now apply the procedure outlined above, including the symmetry reduction, to the Boussinesq equation [10] (BE). This equation is of considerable physical and mathematical interest; for a minireview of its applications and properties, see CK. We thus specify equation (2.1) to be

$$u_{tt}+uu_{xx}+(u_x)^2+u_{xxxx}=0. \tag{2.10}$$

The Lie point symmetry group of (2.10) is well known [6, 11, 12] and consists of dilations and translations, generated by the Lie algebra of vector fields

$$D=x\partial_x+2t\partial_t-2u\partial_u \qquad P_1=\partial_x \qquad P_0=\partial_t. \tag{2.11}$$

The BE is also invariant under discrete transformations generated by time reversal T and coordinate reflection X, defined as

$$T:\quad t\to -t,\ x\to x,\ u\to u \qquad X:\quad t\to t,\ x\to -x,\ u\to u. \tag{2.12}$$

We use a MACSYMA program [5] to perform the operations outlined in (2.5) and (2.6), where Δ^n is now the left-hand side of (2.10). The variables to be eliminated from the resulting expression are u_t (using the condition (2.4)) and u_{xxxx} (using (2.10). The program takes care of all the differential consequences. The program solves part of the determining equations and prints out a reduced system of fourteen determining equations. The first four are easy to solve and from them we find that the general form of the conditional symmetry operator is (2.2) with $\tau(x, t, u)$ arbitrary and

$$\xi=F(x,t)\tau \qquad \phi=(R(x,t)u+S(x,t))\tau. \tag{2.13}$$

Once these partial results are taken into account, the remaining system reduces to five determining equations that can easily be solved by hand. We find that τ remains arbitrary (as it should), and with no loss of generality we can set $\tau=1$.

The resulting conditional symmetry operator is

$$\begin{aligned}V=\partial_t&+[\alpha(t)x+\beta(t)]\partial_x-[2\alpha(t)u+2\alpha(\dot\alpha+2\alpha^2)x^2\\&+2(\alpha\dot\beta+\dot\alpha\beta+4\alpha^2\beta)x+2\beta(\dot\beta+2\alpha\beta)]\partial_u\end{aligned} \tag{2.14}$$

where $\alpha(t)$ and $\beta(t)$ are solutions of the ODE

$$\ddot\alpha+2\alpha\dot\alpha-4\alpha^3=0 \tag{2.15a}$$

$$\ddot\beta+2\alpha\dot\beta-4\alpha^2\beta=0 \tag{2.15b}$$

and the dots denote time derivatives.

These equations are easy to solve and we shall do this below in § 3. Here we shall show that (2.14) and (2.15) already provide the required symmetry reduction and reproduce all the reductions obtained in CK.

The invariants of the local point transformations generated by (2.13) are obtained by solving the PDE

$$V\cdot\Psi(x,t,u)=0.$$

The corresponding characteristic system is

$$\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}x}{\alpha x+\beta}=-\frac{\mathrm{d}u}{2\alpha u+2\alpha(\dot{\alpha}+2\alpha^2)x^2+2(\alpha\dot{\beta}+\dot{\alpha}\beta+4\alpha^2\beta)x+2\beta(\dot{\beta}+2\alpha\beta)}.$$

We solve these two equations to obtain the two invariants z and w, in terms of which we have

$$u(x,t)=w(z)K^2(t)-(\alpha x+\beta)^2 \tag{2.16a}$$

$$z(x,t)=xK(t)-\int_0^t \beta(s)K(s)\,\mathrm{d}s \tag{2.16b}$$

where we have defined

$$K(t)=\exp\left(-\int_0^t \alpha(s)\,\mathrm{d}s\right). \tag{2.16c}$$

Substituting (2.16) into the Boussinesq equation (2.10) we obtain an ODE for $w(z)$:

$$w''''+ww''+w'^2+(Az+B)w'+2Aw=2(Az+B)^2 \tag{2.17}$$

where the primes are z derivatives and

$$A=\frac{\alpha^2-\dot{\alpha}}{K^4}\qquad B=\frac{\alpha\beta-\dot{\beta}}{K^3}+\frac{\alpha^2-\dot{\alpha}}{K^4}\int_0^t \beta(s)K(s)\,\mathrm{d}s. \tag{2.18}$$

Using equations (2.15) for α and β (without necessarily solving them), it is easy to verify that $\mathrm{d}A/\mathrm{d}t=\mathrm{d}B/\mathrm{d}t=0$, i.e. A and B are constants.

Equation (2.17) coincides with equation (3.16) of CK and (2.16) is closely related to equation (3.15) of CK. We have provided the group theoretical origin of the CK reductions and also the corresponding conditional symmetry (2.14) that underlies their reductions.

3. Discussion of the conditional symmetries of the Boussinesq equation

3.1. *The vector fields and the reductions*

In order to make (2.16)-(2.18) explicit we must solve the ODE (2.15). Notice that this is a decomposable system: (2.15*a*) is a simple non-linear equation for α; once α is known, (2.15*b*) is a linear equation for β.

As a matter of fact, (2.15*a*) is a well known equation, namely a special case of the standard Painlevé equation PX in the Painlevé and Gambier classification of second-order differential equations with no movable critical points [13-15]. Its general solution is obtained by putting

$$\alpha=\dot{H}/2H \tag{3.1}$$

where H satisfies

$$\dot{H}^2=h_0H^3+h_1 \qquad h_0, h_1=\text{constant}. \tag{3.2}$$

For $\dot{H}\neq 0$ we obtain β from (2.15*b*) in the form

$$\beta=C_1\frac{\dot{H}}{H}+C_2\frac{\dot{H}}{H}\int_0^t \frac{H(s)}{\dot{H}^2(s)}\,\mathrm{d}s. \tag{3.3}$$

Let us now run through the individual cases.

(a) $h_0 = h_1 = 0$. From (2.15), (2.16)-(2.18) and (3.1) we obtain

$$\alpha = 0 \qquad \beta = \beta_0 + \beta_1 t \qquad K = 1 \qquad A = 0 \qquad B = -\beta_1 \tag{3.4}$$

where β_1 and β_2 are constants.

If $\beta_1 = 0$ we have a classical (translational) symmetry. If $\beta_1 \neq 0$ we use translations in t to set $\beta_0 = 0$, dilations (and possibly time-reversal invariance) to set $\beta_1 = 1$. The resulting conditional symmetry and non-classical reduction are

$$V = \partial_t + t\partial_x - 2t\partial_u \tag{3.5a}$$

$$z = x - \tfrac{1}{2}t^2 \qquad u = w(z) - t^2 \tag{3.5b}$$

$$w''' + ww' - w = 2z + c_1 \tag{3.5c}$$

where c_1 is an integration constant. Equation (3.5c) is obtained from (2.17), using (3.4) with $\beta_0 = 0$, $\beta_1 = 1$ and integrating once. It can be solved in terms of the Painlevé transcendent PII (see CK).

(b) $h_0 \neq 0$, $h_1 = 0$. In this case we find

$$\alpha = -\frac{1}{t} \qquad \beta = \beta_1 t^4 + \frac{\beta_2}{t} \qquad K = t \qquad A = 0 \qquad B = -5\beta_1. \tag{3.6}$$

From (2.16) we see that we can set $\beta_2 = 0$ by translating x. By dilations and coordinate reflections we can transform β_1 into $\beta_1 = 1$, unless we already have $\beta_1 = 0$.

The conditional symmetry and reduction formulae are

$$V = \partial_t + \left(-\frac{x}{t} + \beta_1 t^4\right)\partial_x + \left(\frac{2}{t}u + \frac{6}{t^3}x^2 - 2\beta_1 t^2 x - 4\beta_1^2 t^7\right)\partial_u \qquad \beta_1 = \begin{cases} 0 \\ 1 \end{cases} \tag{3.7a}$$

$$z = xt - \tfrac{1}{6}\beta_1 t^6 \qquad u(x, t) = w(z)t^2 - (x/t - \beta_1 t^4)^2. \tag{3.7b}$$

For $\beta_1 = 0$ the reduced equation can be twice integrated to yield

$$w'' + \tfrac{1}{2}w^2 = c_1 z + c_0 \tag{3.7c}$$

which for $c_1 = 0$ leads to elliptic functions, for $c_1 \neq 0$ to the PI transcendent.

For $\beta_1 = 1$ the reduced equation can be integrated once to give the equation

$$w''' + ww' - 5w = 50z + c_0 \tag{3.7d}$$

that can be solved in terms of the PII transcendent.

(c) $h_0 = 0$, $h_1 \neq 0$. We obtain

$$\alpha = \frac{1}{2t} \qquad \beta = \beta_1 t + \frac{\beta_2}{t} \qquad K = \frac{1}{\sqrt{t}} \qquad A = \tfrac{3}{4} \qquad B = 0. \tag{3.8}$$

Looking at (2.16), or at the conditional symmetry, we see that β_2 can be set equal to zero by x translations and that we either have $\beta_1 = 0$ or β_1 can be dilated and reflected into $\beta_1 = 1$. The case $\beta_1 = 0$ is of no interest here, since it reproduces the classical reduction by dilations. For $\beta_1 = 1$ we have

$$V = \partial_t + \left(\frac{x}{2t} + t\right)\partial_x - \frac{1}{t}(u + 2x + 4t^2)\partial_u \tag{3.9a}$$

$$z = \frac{x}{\sqrt{t}} - \frac{2}{3}t^{3/2} \qquad u = w(z)\frac{1}{t} - \left(\frac{x}{2t} + t\right)^2 \tag{3.9b}$$

$$w'''' + ww'' + (w')^2 + \tfrac{3}{4}zw' + \tfrac{3}{2}w = \tfrac{9}{8}z^2. \tag{3.9c}$$

This is the same equation that is obtained using dilational invariance and has been studied extensively [11, 16, 17].

(*d*) $h_0 \neq 0$, $h_1 \neq 0$. In this case we put $H = P(z)/h_0$ and obtain

$$\alpha = \frac{1}{2}\frac{\dot{P}}{P} \qquad \beta = \beta_1\frac{\dot{P}}{2P} + \beta_2\frac{\dot{P}}{2P}\int_0^t \frac{P(s)\,\mathrm{d}s}{[\dot{P}(s)]^2} \tag{3.10}$$

$$\dot{P}^2 = 4P^3 - g_3 \tag{3.11}$$

where g_3 is a real constant and $P = P(t, 0, g_3)$ is a special case of the Weierstrass elliptic function $P(t, g_2, g_3)$ [18]. The constants g_2 and g_3 are called the invariants of the Weierstrass elliptic functions and are related to the real and imaginary periods. Again, once we write out either the reduction formulae, or the conditional symmetry, we see that β_1 can be set equal to zero by an x translation.

We find

$$K = [P(t)]^{-1/2} \qquad A = -3g_3/4 \qquad B = 0 \tag{3.12}$$

and

$$V = \partial_t + \frac{1}{2}\left(\frac{\dot{P}}{P}x + \beta_2\frac{\dot{P}}{P}W\right)\partial_x$$
$$-\left[\frac{\dot{P}}{P}u + 3\dot{P}x^2 + \frac{\beta_2}{2}\left(\frac{1}{P} + 12\dot{P}W\right)x + \frac{1}{2}\beta_2^2 W\left(\frac{1}{P} + 6\dot{P}W\right)\right]\partial_u \tag{3.13a}$$

$$z = x[P(t)]^{-1/2} + \frac{1}{3}\beta_2 g_3^{-1} P(t)^{-1/2}\int_0^t P(s)\,\mathrm{d}s$$
$$u(x, t) = w(z)P^{-1} - \left(\frac{1}{2}\frac{\dot{P}}{P}x + \beta_2\frac{\dot{P}}{2P}W\right)^2 \tag{3.13b}$$

where we have defined

$$W(t) = \int_0^t \frac{P(s)\,\mathrm{d}s}{[\dot{P}(s)]^2}. \tag{3.13c}$$

The reduced equation in this case is

$$w'''' + ww'' + w'^2 - \tfrac{3}{4}g_3 w' - \tfrac{3}{2}g_3 w = \tfrac{9}{8}g_3^2 z^2. \tag{3.13d}$$

The function $W(t)$ can also be expressed in terms of elliptic integrals, but we shall stick with the Weierstrass elliptic function throughout. The reduction (3.13) is equivalent to the CK reduction in which they use Jacobi elliptic functions and elliptic integrals.

3.2. *Examples of group transformations corresponding to conditional symmetries*

Each of the conditional symmetry operators of § 3.1 can be integrated to give a Lie point transformation, leaving the common solution space of the Boussinesq equation and the corresponding side condition (2.4) invariant.

The transformations are obtained by solving the equations

$$\mathrm{d}x'/\mathrm{d}\lambda = \xi(x', t', u') \qquad \mathrm{d}t'/\mathrm{d}\lambda = \tau(x', t', u') \qquad \mathrm{d}u'/\mathrm{d}\lambda = \phi(x', t', u') \tag{3.14a}$$

subject to the initial conditions

$$(x', t', u')|_{\lambda=0} = (x, t, u). \tag{3.14b}$$

Let us just consider two examples.

(a) Case (3.5). Substituting from (3.5) into (3.14) and integrating we find

$$t' = t + \lambda \qquad x' = x + \lambda t + \tfrac{1}{2}\lambda^2 \qquad u'(x', t') = u(x, t) - 2t\lambda - \lambda^2. \tag{3.15}$$

If we now assume that $u(x, t)$ is a solution of the Boussinesq equation and substitute $u'(x', t')$ into the same equation (for primed variables) we obtain

$$(u_t + tu_x)_x = 0. \tag{3.16}$$

Equation (3.16) is a differential consequence of (2.4), which in this case takes the form

$$u_t + tu_x + 2t = 0. \tag{3.17}$$

In other words, $u'(x', t')$ is a solution of the Boussinesq equation only if $u(x, t)$ satisfies both the Boussinesq equation and (3.16) (which is actually somewhat weaker than (3.17)).

(b) Case (3.7) for $\beta_1 = 0$. The group transformation is

$$t' = t + \lambda \qquad x' = \frac{xt}{t+\lambda} \qquad u'(x', t') = u(x, t)\left(\frac{t+\lambda}{t}\right)^2 + \left(\frac{(t+\lambda)^2}{t^4} - \frac{t^2}{(t+\lambda)^4}\right)x^2. \tag{3.18}$$

Substituting $u'(x', t')$ ito the Boussinesq equation and assuming that $u(x, t)$ is a solution, we obtain a second-order equation for $u(x, t)$. It can be written as

$$A_1(x, t, \lambda)\Delta^1 + A_2(x, t, \lambda)(\Delta^1)_t + A_3(x, t, \lambda)(\Delta^1)_x = 0 \tag{3.19}$$

where

$$\Delta^1 = u_t - \frac{x}{t}u_x - \left(\frac{2}{t}u + 6\frac{x^2}{t^3}\right) \tag{3.20}$$

and the coefficients A_i are

$$A_1 = 2[t^4(t-\lambda) - (t+\lambda)^5] \qquad A_2 = t(t+\lambda)[t^4 - (t+\lambda)^4]$$
$$A_3 = -x[(t+\lambda)^5 - t^4(t-\lambda)]. \tag{3.21}$$

Thus, if $u(x, t)$ satisfies both the Boussinesq equation and the condition $\Delta^1 = 0$, $u'(x', t')$ will also satisfy these conditions (and (3.19) is satisfied).

This example is typical of the general case. For (3.7) with $\beta_1 = 1$, (3.9) and (3.13) we can always quite easily obtain the corresponding group transformation. Substituting $u'(x', t')$ into the Boussinesq equation (of which $u(x, t)$ is a solution) we always obtain an equation of the type (3.19), where $\Delta^1 = 0$ is the appropriate subsidiary condition and the (inessential) coefficients A_1, A_2 and A_3 vary from case to case.

4. Conclusions

The main conclusion that we draw is that, from the point of view of symmetry reduction, the 'non-classical method' [2] should be treated on equal footing with the classical one. Indeed, for the Boussinesq equation, reduction by subgroups of the symmetry group yields similarity solutions (reduction by dilations D), coordinate-independent solutions (reduction by space translations P_1) and a one-parameter family of travelling wave solutions (reduction by $P_0 - vP_1$). The conditional symmetries lead to four individual reductions, namely (3.5), (3.7*c*), (3.7*d*) and (3.9), and to a further one-parameter class of solutions, namely (3.13) (the parameter is g_3).

The 'non-classical method' is actually the classical one, applied not just to the considered equation, but to the equation, together with the condition (2.4). As was emphasised in § 2, the condition (2.4) is different for each conditional symmetry V of (2.2). Moreover, the side condition (2.4) is not necessarily invariant under the symmetry group of the equation under consideration.

Hence conditional symmetries do not form a vector space and cannot be combined with ordinary symmetries.

The procedure that we propose when performing symmetry reduction is the following. First find the symmetry group G of the equation and use it to find all reductions related to invariance under subgroups of G. After that, find all conditional symmetries and use the symmetry group G to simplify them. The first step involves only linear equations, the second step non-linear ones.

Two very important questions remain open. (i) When is the combination of the classical and non-classical method sufficient to find all possible reductions? For the Boussinesq equation Clarkson and Kruskal showed that no other reductions to ODE exist. (ii) When are the conditional symmetries needed? It was shown in CK that for the Burgers, Korteweg-de Vries and modified Korteweg-de Vries equations all reductions are obtained from subgroups of the symmetry group (we have obtained an analogous result for the non-linear Schrödinger equation).

It would be of considerable interest to have an *a priori* criterion telling us when classical symmetries provide all possible reductions.

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References

[1] Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)
[2] Bluman G W and Cole J D 1974 *Similarity Methods for Differential Equations* (Berlin: Springer)
[3] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)
[4] Schwarz F 1985 *Computing* **34** 91
[5] Champagne B and Winternitz P 1985 *Preprint* Montréal CRM-1278
[6] Clarkson P A and Kruskal M D 1989 New similarity solutions of the Boussinesq equation *J. Math. Phys.* in press
[7] Bluman G W and Cole J D 1969 *J. Math. Mech.* **18** 1025

[8] Olver P J and Rosenau Ph 1986 *Phys. Lett.* **114A** 107
[9] Olver P J and Rosenau Ph 1987 *SIAM J. Appl. Math.* **47** 263
[10] Boussinesq J 1871 *C.R. Acad. Sci., Paris* **72** 755
[11] Nishitani T and Tajiri M 1982 *Phys. Lett.* **89A** 379
[12] Rosenau P and Schwarzmeier J L 1986 *Phys. Lett.* **115A** 75
[13] Painlevé P 1900 *Acta Math.* **25** 1
[14] Gambier B 1910 *Acta Math.* **33** 1
[15] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)
[16] Boiti M and Pempinelli F 1980 *Nuovo Cimento* **56B** 148
[17] Quispel G R W, Nijhoff F W and Capel H W 1982 *Phys. Lett.* **91A** 143
[18] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover)